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LOCAL MONODROMY OF p-DIVISIBLE GROUPS

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ABSTRACT. A p-divisible group over a field K admits a slope decomposition; associated to each slope λ is an integer m and a representation $\mathrm{Gal}(K) \to \mathrm{GL}_m(D_\lambda)$, where D_λ is the $\mathbb{Q}_p$-division algebra with Brauer invariant $[\lambda]$. We call m the multiplicity of λ in the p-divisible group. Let G_0 be a p-divisible group over a field k. Suppose that λ is not a slope of G_0, but that there exists a deformation of G in which λ appears with multiplicity one. Assume that $\lambda \neq (s-1)/s$ for any natural number $s > 1$. We show that there exists a deformation G/R of G_0/k such that the representation $\mathrm{Gal}(\mathrm{Frac}\, R) \to \mathrm{GL}_1(D_\lambda)$ has large image.

1. INTRODUCTION

Given a rational number $\lambda \in [0,1]$, where $\lambda = r/s$ with $\gcd(r,s) = 1$, let H_λ be the p-divisible group defined over $\mathbb{F}_p$ whose covariant Dieudonné module is generated by a single generator e satisfying the relation $(F^{s-r} - V^r)e = 0$. The ring of endomorphisms of H_λ which are defined over the algebraic closure of $\mathbb{F}_p$ is an order $\mathcal{O}_{H_\lambda}$ in D_λ, the $\mathbb{Q}_p$-division algebra whose Brauer invariant is the class of λ in $\mathbb{Q}/\mathbb{Z}$. By a theorem of Dieudonné and Manin a p-divisible group G over a field K is isogenous, over the algebraic closure $\overline{K}$ of K, to a sum $\oplus_{\lambda \in \mathbb{Q}} H_{\lambda,\overline{K}}^{m_\lambda}$ [9]. The numbers m_λ are uniquely determined by G, and we say that the slope λ appears in G with multiplicity m_λ.

Following Gross [5], we can associate to G the $\mathrm{Gal}(K)$ module $V^\lambda(G) := \mathrm{Hom}(H_{\lambda,\overline{K}}, G_{\overline{K}}) \otimes \mathbb{Q}$. This yields (see Definition 2.2) a representation

$$\rho^\lambda = \rho : \mathrm{Gal}(K) \longrightarrow \mathrm{GL}_{m_\lambda}(D_\lambda), \tag{1.1}$$

which we call the λ-monodromy of G.

We say (Definition 2.3) that the λ-monodromy is large if there is a subgroup of $\mathrm{GL}_{m_\lambda}(D_\lambda)$ that has finite index in both $\mathrm{GL}_{m_\lambda}(\mathcal{O}_{H_\lambda})$ and $\rho(\mathrm{Gal}(K))$.

Let k be an algebraically closed field of characteristic p and let R be an equicharacteristic complete local domain with residue field k and fraction field K. Let G_0 be a p-divisible group over k and G a lifting to R. In these circumstances the representation ρ has been studied extensively. The first major work was done by Igusa [7]. Assume that G_0 is the p-divisible group of a supersingular elliptic curve and that G_K has slopes 0 and 1. Then the action of $\mathrm{Gal}(K)$ on $\mathrm{Hom}(H_0, G_K) \cong \mathbb{Z}_p$ is cofinite in $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$. Both Gross [5] and Chai [1] studied the case when G_0 has a single slope $c/(c+1)$ and G_K has slope $g/(g+1)$ with multiplicity one and slope 1 with multiplicity $c - g$. Gross showed that the image of $\mathrm{Gal}(K)$ in

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$\mathrm{Aut}(H_{(g-1)/g})$ is all of $\mathrm{Aut}(H_{(g-1)/g})$, while Chai showed that the slope-1 representation acting on $\mathbb{Q}_p^{(c-g)}$ is irreducible. We make no attempt at a comprehensive history of this problem, but do note that a global analogue of this question for the generic Newton stratum of certain moduli spaces of PEL type has been resolved by Deligne and Ribet [3] and Hida [6].

Definition 1.1. Let G_0 be a p-divisible group over an algebraically closed field k. We say a slope λ is *attainable* from G_0 if there exists a deformation G/R of G_0 to a complete local domain R with fraction field K such that λ appears as a slope of G_K with multiplicity one; and, furthermore, we require that if λ' is a slope of G_K with $\lambda' < \lambda$, then λ' appears in G_0 with the same multiplicity as in G_K. We say that such a G attains λ.

(The λ which are attainable from G_0 are determined completely by the Newton polygon of G_0, see Theorem 2.1.)

Our main result is:

Theorem 1.2. *Let G_0 be a p-divisible group over an algebraically closed field of characteristic p. Assume that a rational number $\lambda \in [0,1]$ is not a slope of G_0, that $\lambda \neq (s-1)/s$ for any natural number $s \geq 2$, and that λ is attainable from G_0. Then there exists a deformation of G_0 which attains λ and has large λ-monodromy.*

For applications to families of abelian varieties, we need a variant of 1.2 adapted to deformations of p-divisible groups equipped with quasi-polarizations. A principal quasi-polarization of a p-divisible group is a self-dual isomorphism $\Phi : G \to G^t$.

Definition 1.3. Given a principally quasi-polarized (or pqp for short) p-divisible group $(G_0/k, \Phi_0)$, k algebraically closed, we say a rational number $\lambda \in [0,1]$ is symmetrically attainable from (G_0, Φ_0) if there is a pqp deformation of (G_0, Φ_0) to a pqp p-divisible group (G, Φ) over a complete local domain R such that G attains λ. In this case we say that (G, Φ) symmetrically attains λ.

Theorem 1.4. *Let (G_0, Φ_0) be a pqp p-divisible group over an algebraically closed field of characteristic p. Assume that λ is not a slope of G_0, that $\lambda \neq (s-1)/s$ for any natural number $s \geq 2$, and that λ is symmetrically attainable from (G_0, Φ_0). Then there exists a pqp deformation of (G_0, Φ_0) to (G, Φ) over a complete local domain R with fraction field K so that G_K symmetrically attains λ and has large λ-monodromy.*

We begin by reviewing some facts in section two about p-divisible groups, their automorphisms, Newton polygons, and monodromy. The heart of the paper is section three. There we consider a local p-divisible group G_0 over an algebraically closed field k with a-number equal to one, i.e., $\dim(\mathrm{Hom}_k(\boldsymbol{\alpha}_p, G_0)) = 1$. We also assume that λ is positive and is strictly less than any slope of G_0. We construct, using techniques of Oort, and then analyze a particular deformation of G_0 that attains λ with multiplicity one and show it has large λ-monodromy. In section four we prove two reduction steps that allow us to complete the proof of Theorem 1.2 for positive slopes, and separately we prove the case of slope zero. The last section is devoted to proving Theorem 1.4, the analogue of Theorem 1.2 for principally quasi-polarized p-divisible groups.

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2. BACKGROUND ON p-DIVISIBLE GROUPS

2.1. **Slopes, the slope filtration and Newton polygons.** We describe the Newton polygon of a p-divisible group over a field. The Newton polygon determines and is determined by the isogeny class of a p-divisible group over an algebraically closed field.

Let G be a p-divisible group over a field K of characteristic p. By a theorem of Grothendieck (proved in [14]), G has a filtration

$$0 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G \tag{2.1}$$

by p-divisible groups so that G_i/G_{i-1} is a p-divisible group isogenous over the algebraic closure of K to the direct sum of m_{λ_i} copies of H_{λ_i}, and the rational numbers λ_i satisfy $\lambda_i < \lambda_{i+1}$. Write $\lambda_i = r_i/s_i$ where $\gcd(r_i, s_i) = 1$. Then the height of successive steps in the filtration is $\operatorname{height}(G_i/G_{i-1}) = m_{\lambda_i}s_i$. By slope we mean the slope of the Frobenius acting on the covariant Dieudonné module, which corresponds to the Verschiebung operator of a p-divisible group. For example, $\boldsymbol{\mu}_{p^\infty}$ has slope zero.

The Newton polygon $\operatorname{NP}(G)$ of G/K is the convex hull of the set

$$\{(0,0)\} \cup_{i=1}^{n} \{(\operatorname{height}(G_i), \sum_{j=1}^{i} \operatorname{height}(G_i/G_{i-1})\lambda_i)\} \subset \mathbb{Z}^2 \subset \mathbb{R}^2.$$

It is a lower-convex polygon connecting $(0,0)$ to $(\operatorname{height}(G), \dim(G))$ with slopes in the interval $[0,1]$ and integral breakpoints. Any such Newton polygon is actually realized as the Newton polygon of a p-divisible group.

Let ν_1 and ν_2 be two Newton polygons. Following Oort we say that $\nu_1 \succeq \nu_2$ if ν_1 and ν_2 share the same endpoints and if every point of ν_1 is on or below that of ν_2.

Let G_0/k be a p-divisible group over an algebraically closed field k. By a deformation of G_0 we mean a p-divisible group G over a local ring R equipped with isomorphisms $R/\mathfrak{m}_R \cong k$ and $G \times R/\mathfrak{m}_R \cong G_0/k$.

Grothendieck [8, Theorem 2.1.3] proved that the Newton polygon goes up under specialization and conjectured, conversely, that one can always achieve an arbitrary "lower" Newton polygon by deformation. Oort [10] proved this converse, and we make use of his ideas throughout this paper.

Theorem 2.1. [10] *Let G_0 be a p-divisible group over an algebraically closed field k. Let ν be an arbitrary Newton polygon. Then there exists a deformation G/R of G_0 such that $\operatorname{NP}(G \times \operatorname{Frac} R) = \nu$ if and only if $\operatorname{NP}(G_0) \preceq \nu$.*

There is a variant of this theory for principally quasi-polarized p-divisible groups. A Newton polygon is called symmetric if, in the notation introduced above, $\lambda_i = 1 - \lambda_{n+1-i}$ and $m_{\lambda_i} = m_{\lambda_{n+1-i}}$. Much in the vein of Theorem 2.1 Oort proves that if G_0 is a principally quasi-polarized p-divisible group, and if ν is a symmetric Newton polygon with $\operatorname{NP}(G_0) \preceq \nu$, then there exists a principally quasi-polarized deformation of G_0 with generic Newton polygon ν.

2.2. **Monodromy of p-divisible groups.** As in the introduction, let $H_{r/s}$ be the p-divisible group with Dieudonné module $F^{s-r} = V^r$. Dieudonné and Manin have shown that if G is a p-divisible group over an algebraically closed field k, then

there exists an isogeny

$$\oplus_{\lambda\in\mathbb{Q}} H_\lambda^{\oplus m_\lambda} \longrightarrow G. \tag{2.2}$$

Now suppose G is a p-divisible group over an arbitrary field K of characteristic p. (For ease of exposition below, we will always assume that K contains the algebraic closure $\overline{\mathbb{F}}_p$ of the prime field.) In general, the slope filtration (2.1) only splits, even up to isogeny, after passage to the perfect closure K^{perf} of K [14]. Moreover, even if K is perfect, an isogeny (2.2) need not exist over K. The field of definition of such an isogeny is a measure of the complexity of the p-divisible group. Henceforth let $\mathrm{Gal}(K) = \mathrm{Gal}(\overline{K}/K^{\mathrm{perf}})$; it is canonically isomorphic to $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Let $\lambda = \lambda_i$ be one of the slopes of G/K, in the sense that $m_\lambda > 0$. Then $\mathrm{Aut}(H_\lambda)$ acts on $V^\lambda = \mathrm{Hom}(H_\lambda, (G_i/G_{i+1})) \otimes \mathbb{Q}$ on the right. Moreover, $\mathrm{Gal}(K)$ acts on V^λ on the left; $\tau \in \mathrm{Gal}(K)$ takes $f \in \mathrm{Hom}_{\overline{K}}(H_{\lambda,\overline{K}}, (G_i/G_{i-1})_{\overline{K}})$ to $\tau \circ f \circ \tau^{-1}$. The actions of $\mathrm{Gal}(K)$ and D_λ commute, and we obtain a representation $\rho : \mathrm{Gal}(K) \to \mathrm{Aut}_{D_\lambda}(V^\lambda)$.

Definition 2.2. We call this the λ-monodromy of G, and the image of $\mathrm{Gal}(K)$ in $\mathrm{Aut}((G_i/G_{i-1})_{\overline{K}})$ the λ-monodromy group of G. If R is a complete local domain and G/R is a p-divisible group, the λ-monodromy of G/R is that of $G \times \mathrm{Frac}\, R$.

Given a choice of isomorphism $V^\lambda \to D_\lambda^{m_\lambda}$ we obtain a representation in $\mathrm{GL}_{m_\lambda}(D_\lambda)$. Our goal is to show that the monodromy differs little from $\mathrm{GL}_{m_{\lambda_i}}(\mathcal{O}_\lambda)$. We say that two subgroups of $\mathrm{GL}_{m_\lambda}(D_\lambda)$ are commensurable if there is a single subgroup of finite index in both groups.

Definition 2.3. We call a subgroup of $\mathrm{GL}_{m_\lambda}(D_\lambda)$ *large* if it is commensurable with $\mathrm{GL}_{m_\lambda}(\mathcal{O}_{H_\lambda})$.

Even though the λ-monodromy group depends on the choice of isogeny (2.2), of isomorphism $V^\lambda \to D_\lambda^{m_\lambda}$, and of slope-$\lambda$ test object, we will see below that having large λ-monodromy is independent of all of these choices.

It will often be more convenient to calculate monodromy in the category of F-lattices. Assume K is a perfect field of characteristic p and let σ denote the Frobenius on $W(K)$. By an F-lattice we mean a free, finitely generated $W(K)$-module with an injective σ-linear operator F. A Dieudonné module over K gives us an F-lattice by forgetting the action of V. We say that F-lattices M_1, M_2 are isogenous if there is an F-equivariant map $M_1 \to M_2$ with $W(K)$-torsion kernel and cokernel. If M_1, M_2 are two Dieudonné modules over K and if $\widetilde{M}_i$ is an F-lattice which is isogenous to M_i as F-lattice for $i = 1, 2$, then

$$\mathrm{Hom}_{\mathbb{D}}(M_1, M_2) \otimes \mathbb{Q} \cong \mathrm{Hom}_F(\widetilde{M}_1, \widetilde{M}_2) \otimes \mathbb{Q},$$

where the left-hand side denotes homomorphisms of Dieudonné modules and the right-hand side means homomorphisms of F-lattices. Somewhat more precisely, we have [4, IV.1]:

Lemma 2.4. *Let M_1 and M_2 be Dieudonné modules over a perfect field K which are isogenous as F-lattices. Then M_1 and M_2 are isogenous as Dieudonné modules, and* $\mathrm{Hom}_{\mathbb{D}}(M_1, M_2)$ *has finite index in* $\mathrm{Hom}_F(M_1, M_2)$.

Therefore, in order to compute λ-monodromy, we may work in the category of F-lattices, rather than the category of Dieudonné modules. If M and N are two

F-lattices, we say two subgroups of $\mathrm{Hom}_F(M,N)\otimes\mathbb{Q}$ are commensurable if there is a single subgroup of finite index in both.

Lemma 2.5. *For $i=1,2$, let M_i and N_i be F-lattices over K. If M_1 and M_2 are isogenous, and if N_1 and N_2 are isogenous, then $\mathrm{Hom}_F(M_1,N_1)$ and $\mathrm{Hom}_F(M_2,N_2)$ are commensurable, as are $\mathrm{Aut}_F(M_1)$ and $\mathrm{Aut}_F(M_2)$.*

Proof. If M and N are F-lattices, then $\mathrm{Hom}(M,N)$ is naturally a summand of $\mathrm{End}(M\oplus N)$. Therefore, for the first claim it suffices to prove that, for any pair of F-lattices M_1 and M_2, an isogeny $\phi: M_1\to M_2$ identifies an open subgroup of $\mathrm{End}(M_1)$ with an open subgroup of $\mathrm{End}(M_2)$.

Now, ϕ induces an isomorphism $M_1\otimes\mathbb{Q}\cong M_2\otimes\mathbb{Q}\cong V$; we view $\mathcal{E}_i:=\mathrm{End}(M_i)$ as a subgroup of $\mathrm{End}(V)$.

Let $\mathcal{E}_i(n)=p^n\mathcal{E}_i=\{\alpha\in\mathrm{End}(M_i):\alpha(M_i)\subseteq p^nM_i\}$. Each $\mathcal{E}_i(n)$ has finite index in $\mathcal{E}_i$.

Suppose that $p^nM_2\subset M_1\subset p^{-n}M_2$. Then $\alpha\in\mathcal{E}_2(2n)$ maps M_1 to itself. Therefore, $\mathcal{E}_2(2n)\subset\mathcal{E}_1$. In particular, $\mathcal{E}_2(2n)\subset\mathcal{E}_1\cap\mathcal{E}_2\subset\mathcal{E}_2$, so that $\mathcal{E}_1\cap\mathcal{E}_2$ has finite index in $\mathcal{E}_2$. After taking an isogeny $M_2\to M_1$, we similarly see that $\mathcal{E}_1\cap\mathcal{E}_2$ has finite index in $\mathcal{E}_1$.

To prove the second claim we use $\phi: M_1\to M_2$ to view $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$ as subgroups of $\mathrm{Aut}(M_1\otimes\mathbb{Q})$. Let $A=\{g\in\mathrm{Aut}(M_1)\mid(g-1)M_1\subseteq p^nM_1\}$ for sufficiently large n. Then $A\subseteq\mathrm{Aut}(M_1)\cap\mathrm{Aut}(M_2)$, and A has finite index in both $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$. □

Below, for $\lambda=r/s$ we will need to consider the F-lattice

$$N_{\lambda,F}=W(\mathbb{F}_p)[p^{1/s}][F]/(F-p^{\lambda}). \tag{2.3}$$

This F-lattice is isogenous to the F-lattice obtained from the Dieudonné module of H_λ. Note that $N_{\lambda,F}$ admits an endomorphism ϖ that maps (the equivalence class of) 1 to $p^{1/s}$. For a field K containing $\mathbb{F}_q$, with $q=p^s$, the endomorphism ring of $N_{\lambda,F}(K)$ is $\mathcal{O}_\lambda=W(\mathbb{F}_q)[\varpi]$, an order in the division algebra over $\mathbb{Q}_p$ with Brauer invariant λ. Note that $\varpi^s=p$, and that if $x\in W(\mathbb{F}_q)\subset\mathcal{O}_\lambda$ then $x\varpi=\varpi x^\tau$, where $\tau=\sigma^r\in\mathrm{Aut}(W(\mathbb{F}_q)/\mathbb{Z}_p)$. In fact, since $\gcd(r,s)=1$, τ is a generator of $\mathrm{Aut}(W(\mathbb{F}_q)/\mathbb{Z}_p)$. The automorphism group of $N_{\lambda,F}$ is

$$\mathcal{G}_\lambda=W(\mathbb{F}_q)[\varpi]^\times; \tag{2.4}$$

we consider the structure of G_λ in Section 3.2.

3. A SPECIAL CASE

In this section we prove a special, but crucial, case of our main result. Let G_0 be a local p-divisible group over an algebraically closed field k. We assume that the a-number of G_0 is one, that $\lambda=r/s$ is positive and less than any slope of G_0, that $\lambda\neq(s-1)/s$ for any natural number s, and that λ is attainable from G_0. Let $G^{\mathrm{univ}}/R^{\mathrm{univ}}$ be the universal deformation of G_0 over the universal deformation ring R^{univ}. Given a Newton polygon ν with $\nu\succeq\mathrm{NP}(G_0)$, by [8] there is a radical ideal $J=J_\nu$ of R^{univ} characterized as follows: if R_1 is a complete local domain and G_{R_1} a deformation of G_0 whose Newton polygon is on or above ν, then the deformation G_{R_1} is induced by a map $R^{\mathrm{univ}}/J\to R_1$; and if $x\in\mathrm{Spec}(R^{\mathrm{univ}}/J)$, then $\mathrm{NP}(G_x)\preceq\nu$. Let $\nu=\mathrm{NP}(*)$, the Newton polygon obtained by adjoining

(s,r) to the Newton polygon of G_0. For this choice of ν set $R = R^{\mathrm{univ}}/J$ and $G = G_R^{\mathrm{univ}}$. (We given an explicit description of R in (3.10).)

Lemma 3.1. *Let G_0/k be a local p-divisible group over an algebraically closed field, with a-number $a(G_0) := \dim_k \operatorname{Hom}(\boldsymbol{\alpha}_p, G_0) = 1$. Suppose that λ is a positive rational number strictly smaller than any slope of G_0, that $\lambda \neq (s-1)/s$ for any natural number s, and that λ is attainable from G_0. Let G_R be the deformation described above. Then the deformation G_R of G_0 has large λ-monodromy.*

(Note that by hypothesis $0 < \lambda < 1$ and $s \geq 3$.) Let K denote the fraction field of R. The hypotheses on G_0 and λ force $a(G_K) = 1$. Therefore, much information about G_K is encoded in the (noncommutative) characteristic polynomial of its Frobenius operator. In Section 3.1, we use a result of Demazure on factorization of such polynomials to give a method for computing the lowest-slope monodromy of a p-divisible group. In Section 3.2 we collect some remarks on the structure of G_λ, the ambient group for the λ-monodromy group.

After reviewing how deformations of G_0 are described by displays (Section 3.3), we analyze the deformation $G = G_R^{\mathrm{univ}}$ of G_0 in which λ appears with multiplicity one (Section 3.4). We use the method of Demazure to explicitly show that certain graded pieces of the λ-monodromy of G are maximal, and thus that the λ-monodromy of G is large.

3.1. **A Lemma of Demazure.** In his book on p-divisible groups, Demazure proves the following result [4, Lemma IV.4.2] about polynomials over the noncommutative ring $W(K)[F]$, where $Fa = a^\sigma F$ for $a \in W(K)$.

Lemma 3.2. *Given a polynomial in $W(K)[F]$,*

$$\chi(F) = F^n + A_1F^{n-1} + \cdots + A_n, \tag{3.1}$$

let $\lambda = \frac{r}{s} = \min_i(\frac{\operatorname{ord}_p A_i}{i})$ with $\gcd(r,s) = 1$. Over $W(\overline{K})[p^{\frac{1}{s}}]$ there exists a factorization

$$\begin{aligned}\chi(F) &= \chi_1(F)\chi_2(F)\\ &= \chi_1(F)\cdot(F - p^\lambda)u,\end{aligned}$$

where the element $v = u^{-1} \in W(\overline{K})[p^{\frac{1}{s}}]$ satisfies

$$v^{\sigma^n} + a_1 v^{\sigma^{n-1}} + \cdots + a_n = 0,$$

with $a_i = A_i p^{-i\lambda}$.

In the situation where λ is the smallest slope of M and occurs with multiplicity one, we can exploit such a factorization to compute the monodromy of M in slope λ as the Galois group of an equation such as (3.1). If $u \in W(\overline{K})$, by $K(u)$ we mean the extension of K generated by all Witt components of u. Similarly, if $u \in W(\overline{K})[p^{1/s}]$, then u may be written as $u = \sum_{i=0}^{s-1} u_i p^{i/s}$, and by $K(u)$ we mean the extension of K generated by all Witt components of each of the u_i.

Lemma 3.3. *Let M be a Dieudonné module over a perfect field K generated by a single element e that satisfies*

$$\chi(F)e = (F^n + A_1F^{n-1} + \cdots + A_n)e = 0,$$

where n is the rank of M as $W(K)$-module. Suppose that the slope $\lambda = \min_i(\frac{\operatorname{ord}_p A_i}{i}) = r/s$ occurs in M with multiplicity one. Set $a_i = p^{-\lambda i} A_i$. Then the monodromy group of M in slope λ is induced by the action of $\operatorname{Gal}(K(v)/K)$ on $u = v^{-1}$, where $v \in W(\overline{K})[p^{1/s}]$ satisfies

$$v^{\sigma^n} + a_1 v^{\sigma^{n-1}} + \cdots + a_n = 0. \tag{3.2}$$

Proof. By Lemma 2.4, it suffices to prove the result for F-lattices, rather than for Dieudonné modules. As F-lattice, M is isogenous to the F-lattice $M_\chi = W(K)[F]/\chi(F)$. Let $N_\lambda = \mathbb{D}_*(H_\lambda)$ be the Dieudonné module of slope λ defined in the introduction. As F-lattice $N_{\lambda,F}$ is isogenous to N_λ, so we can replace the computation of $\operatorname{Hom}_{\mathbb{D}}(N_\lambda(\overline{K}), M(\overline{K}))$ with the calculation of $\operatorname{Hom}_F(N_{\lambda,F}(\overline{K}), M_\chi(\overline{K}))$. Since M_χ is defined over a perfect field it is isogenous to the direct sum of two lattices $M_\chi \sim M'_\chi \oplus M''_\chi$, where the only slope of M'_χ is λ, with multiplicity one, and where λ is not a slope of M''_χ. Thus we can replace our calculation with the calculation of $\operatorname{Hom}_F(N_{\lambda,F}(\overline{K}), M'_\chi(\overline{K}))$.

Our next step is to use Demazure's lemma 3.2 to make explicit the action of $\operatorname{Gal}(K)$ on $M'_\chi(\overline{K})$. This lemma gives us a map of F-lattices

$$M_\chi(\overline{K}) = W(\overline{K})[F]/\chi(F) \xrightarrow{\psi} M_\lambda(\overline{K}) = W(\overline{K})[p^{1/s}][F]/(F - p^\lambda)u.$$

Denote the implicitly given generator of M_χ by e_χ and set $e_{M_\lambda} = \psi(e_\chi)$. For $x \in W(\overline{K})$ and $g \in \operatorname{Gal}(K)$ we have $(xe_\chi)^g = x^g e_\chi$. We claim that $\ker\psi$ is Galois invariant, and thus $(xe_{M_\lambda})^g = x^g e_{M_\lambda}$. The map ψ induces a diagram of quasi-morphisms

$$\begin{tikzcd}
M_\chi(\overline{K}) \arrow[r] \arrow[d] & M_\lambda(\overline{K}) \arrow[d] \\
M'_\chi(\overline{K}) \oplus M''_\chi(\overline{K}) \arrow[r, "\widetilde{\psi}"] & M_\chi(\overline{K})
\end{tikzcd}$$

where the vertical maps are quasi-isogenies. Since $\ker\widetilde{\psi}$ is Galois invariant, so is $\ker\psi$.

Let e_{N_λ} denote the implicitly given generator of $N_{\lambda,F}$. Then

$$\begin{aligned} N_\lambda &\longrightarrow M_\lambda(\overline{K}) \\ e_{N_\lambda} &\longmapsto u e_{M_\lambda} \end{aligned}$$

is an isomorphism of F-lattices over $\overline{K}$. The action of $\operatorname{Gal}(K)$ on $\operatorname{Hom}_F(N_\lambda, M)$ is therefore the action of $\operatorname{Gal}(K)$ on u, so that $\operatorname{Gal}(K(u)/K)$ induces an action commensurable with the slope λ-monodromy of M.

$\square$

Choosing an isomorphism $M_\lambda(\overline{K}) \cong N_{\lambda,F}$ allows us to view the monodromy representation as a map from $\operatorname{Gal}(K)$ to the group $\mathcal{G}_\lambda$ defined in (2.4). We next make some preparatory remarks on the structure of $\mathcal{G}_\lambda$.

3.2. **Subgroups of** $\mathrm{Aut}(N_{\lambda,F})$**.** We continue to let $\lambda = r/s$, where $r \geq 1$ and $\gcd(r,s) = 1$. While the hypotheses of Lemma 3.1 imply that $s \geq 3$, the results of the present subsection are valid for $s = 2$, too. Let $q = p^s$. Recall (2.4) that $\mathcal{O}_\lambda = \mathrm{End}(N_{\lambda,F})$ has a presentation as $W(\mathbb{F}_q)[\varpi]$, where $\varpi^s = p$ and there is a generator τ of $\mathrm{Aut}(W(\mathbb{F}_q)/\mathbb{Z}_p)$ such that, for all $x \in W(\mathbb{F}_q) \subset \mathcal{O}_\lambda$, $x\varpi = \varpi x^\tau$. Let $\mathcal{G} = \mathcal{G}_0 = \mathcal{O}_\lambda^\times$, and let $\mathcal{G}_i = 1 + \varpi^i \mathcal{O}_\lambda \subset \mathcal{G}$. For any subgroup $\mathcal{H} \subset \mathcal{G}$, let $\mathcal{H}_i = \mathcal{H} \cap \mathcal{G}_i$. There is always a natural inclusion

$$\mathcal{H}_i/\mathcal{H}_{i+1} \hookrightarrow \mathcal{G}_i/\mathcal{G}_{i+1} \cong \begin{cases} \mathbb{F}_q^\times & i = 0 \\ \mathbb{F}_q^+ & i \geq 1 \end{cases}. \tag{3.3}$$

In Section 3.4, we will construct a p-divisible group whose λ-monodromy $\mathcal{H}$ satisfies $\mathcal{H}_i/\mathcal{H}_{i+1} = \mathcal{G}_i/\mathcal{G}_{i+1}$ for $i = 0$, 1 and s. We will show (Lemma 3.5) that such a group is in fact equal to $\mathcal{G}$.

The structure of an order in a division algebra over a local field is efficiently documented in [12]. We make the isomorphisms in (3.3) explicit now. An element $\alpha \in W(\mathbb{F}_q)$ can be written as

$$\alpha = \sum_{j \geq 0} p^j \langle \alpha_j \rangle,$$

where $\alpha_j \in \mathbb{F}_q$ and $\langle \alpha_j \rangle = (\alpha_j, 0, \cdots, 0) \in W(\mathbb{F}_q)$. Now, the order $\mathcal{O}_\lambda$ is a finite, free $W(\mathbb{F}_q)$-module; any $\beta \in \mathcal{O}_\lambda$ has a unique expression

$$\beta = \sum_{j=0}^{s-1} \varpi^j \beta_j,$$

with $\beta_j \in W(\mathbb{F}_q)$. Taking the expansion of each β_j as above and relabeling, we have

$$\beta = \sum_{j \geq 0} \varpi^j \langle \beta_j \rangle.$$

Lemma 3.4. *Suppose that* $\mathcal{H}_s/\mathcal{H}_{s+1} = \mathcal{G}_s/\mathcal{G}_{s+1}$. *Then* $\mathcal{H}_{ns}/\mathcal{H}_{ns+1} = \mathcal{G}_{ns}/\mathcal{G}_{ns+1} \cong \mathbb{F}_q$ *for all natural numbers* n*.*

Proof. The proof is by induction on n. Each class in $\mathcal{G}_{(n+1)s}/\mathcal{G}_{(n+1)s+1}$ is represented by $1 + \varpi^{(n+1)s}\langle \alpha \rangle$ for some $\alpha \in \mathbb{F}_q$. By the inductive hypothesis, there exists an element $1 + \varpi^{ns}\langle \alpha \rangle + \varpi^{ns+1}\widetilde{\beta} \in \mathcal{H}_{ns}$ for some $\widetilde{\beta} \in \mathcal{O}_\lambda$. Then

$$\begin{aligned}(1 + \varpi^{ns}\langle \alpha \rangle + \varpi^{ns+1}\widetilde{\beta})^p &= 1 + p(\varpi^{ns}\langle \alpha \rangle + \varpi^{ns+1}\widetilde{\beta}) + p^2 \varpi^{ns}\widetilde{\gamma} \\ &\equiv 1 + \varpi^{(n+1)s}\langle \alpha \rangle \bmod \mathcal{G}_{(n+1)s+1}\end{aligned}$$

for some $\widetilde{\gamma} \in \mathcal{O}_\lambda$, since $\varpi^s = p$. □

Lemma 3.5. *Suppose that* $\mathcal{H}_i/\mathcal{H}_{i+1} = \mathcal{G}_i/\mathcal{G}_{i+1}$ *for* $i = 0, 1$ *and* s*. Then* $\mathcal{H}/\mathcal{H}_n = \mathcal{G}/\mathcal{G}_n$ *for all natural numbers* n*.*

Proof. It suffices to show that for all $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}_n/\mathcal{H}_{n+1} = \mathcal{G}_n/\mathcal{G}_{n+1}$. Again, we prove this by induction on n. If $s|(n+1)$, then $\mathcal{H}_{n+1}/\mathcal{H}_{n+2} \cong \mathcal{G}_{n+1}/\mathcal{G}_{n+2}$ by Lemma 3.4. Otherwise, by induction we may assume that $\mathcal{H}_i/\mathcal{H}_{i+1} = \mathcal{G}_i/\mathcal{G}_{i+1}$ for $i = 1$ and $i = n$. A direct computation [12, Lemma 1.1.8] shows that $[1 - \varpi\langle x \rangle, 1 - \varpi^n\langle y \rangle] \equiv 1 + \varpi^{n+1}(x^{\tau^n}y - y^\tau x) \bmod \mathcal{G}_{n+2}$. Since $s \nmid (n+1)$, *every* element of $\mathbb{F}_q$ is of the form $x^{\tau^n}y - y^\tau x$; the result follows. □

3.3. **Displays.** Throughout this section we use the ideas, results, and terminology of Oort's paper [10]. If R is a ring of positive characteristic, and if $x \in R$, let $\langle x \rangle = (x, 0, 0, \cdots) \in W(R)$.

Let k be a perfect field and M_0 a Dieudonné module over k. Denote the dimension of M_0 by d, the codimension by c, and the height by $h = d + c$. By a display of M_0 we mean a choice of $W(k)$-basis of M_0, $\{e_i : 1 \leq i \leq h\}$, along with relations defining M_0:

$$Fe_i = \sum a_{ji} e_j \qquad 1 \leq i \leq d \tag{3.4}$$

$$e_i = V(\sum a_{ji} e_j) \qquad d+1 \leq i \leq h \tag{3.5}$$

We often summarize this data in the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.6}$$

where $A = (a_{ij})_{1 \leq i \leq d, 1 \leq j \leq d}$, $B = (a_{ij})_{1 \leq i \leq d, d+1 \leq j \leq h}$, $C = (a_{ij})_{d+1 \leq i \leq h, 1 \leq j \leq d}$, and $D = (a_{ij})_{d+1 \leq i \leq h, d+1 \leq j \leq h}$.

We define certain subsets of $\mathbb{Z} \times \mathbb{Z}$:

$$\begin{aligned} S &= \{(i,j) : 1 \leq i \leq d, d \leq j \leq h\} \\ S^{\text{univ}} &= \{(i,j) : 1 \leq i \leq d, d+1 \leq j \leq h\}. \end{aligned}$$

The universal equicharacteristic deformation of M_0 is defined over the ring

$$R^{\text{univ}} := k[\![t_{ij} : (i,j) \in S^{\text{univ}}]\!]. \tag{3.7}$$

Let T be the $d \times c$ matrix with entries $T_{ij} = \langle t_{ij} \rangle$. Then the universal deformation of M_0 is displayed by

$$\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}. \tag{3.8}$$

(The theory of displays over rings which are not necessarily perfect is documented in [13].)

If the a-number of M_0 is one, then one can choose a basis for M_0 so that the display (3.4) becomes particularly simple. Indeed, if the a-number of M_0 is one, so $\dim_k(M_0/(F,V)M_0)$ is one, there exists [10, 2.2] a display so that the matrix (a_{ij}) has the form

$$\left(\begin{array}{ccccc|ccccc}
0 & & \cdots & 0 & a_{1,d} & a_{1,d+1} & & \cdots & & a_{1,h} \\
1 & 0 & \cdots & 0 & a_{2,d} & a_{2,d+1} & & \cdots & & \\
0 & 1 & \cdots & & & \vdots & & & & \\
\vdots & & & & \vdots & & & & & \\
0 & \cdots & & 1 & a_{d,d} & \cdots & & & & a_{d,h} \\
\hline
0 & \cdots & & 1 & & 0 & & \cdots & & 0 \\
\vdots & & & 0 & & 1 & 0 & \cdots & & 0 \\
 & & & & & 0 & 1 & \cdots & & \\
\vdots & & & \vdots & & \vdots & & & & \vdots \\
0 & \cdots & & 0 & & 0 & \cdots & & 1 & 0
\end{array}\right)$$

with $a_{1,h}$ a unit in $W(k)$. We call such a display a normal form for M_0. In this case, Oort shows there is an explicit polynomial $\chi_0(F)$ so that the generator e of

M_0 satisfies

$$\chi_0(F)e = (F^h - \sum_{x=0}^{h-1} A_x F^{h-x})e = 0$$

with $A_x \in W(k)$. This polynomial depends on the choice of normal display. In spite of this ambiguity we call $\chi_0(F)$ the characteristic polynomial of M_0. Oort shows that the Newton polygon of M_0 equals the Newton polygon of χ_0. We write out the formula for the coefficient A_x in terms of the entries a_{ij} of the normal display, where $(i, j) \in S$.

Define a map

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{f=(x,y)} & \mathbb{Z}^2 \\ & & \\ (i,j) & \longmapsto & (x(i,j), y(i,j)) = (j+1-i, j-d). \end{array}$$

With this notation, Oort's Cayley-Hamilton lemma [10, 2.6] says that

$$A_x = \sum_{(i,j):(i,j)\in S, x(i,j)=x} p^{y(i,j)} a_{ij}^{\sigma^{h-y(i,j)-d}}.$$

Observe that, since σ is additive, this formula is additive in a_{ij}.

The display for the universal deformation M^{univ} is in normal form, and hence it is determined by the entries in the positions $(i, j) \in S$. Let δ be the translation map $\delta(i, j) = (i, j+1)$. A display in normal form for M^{univ} is given by the matrix (a_{ij}^{univ}), where

$$a_{ij}^{\mathrm{univ}} = \begin{cases} a_{ij} + \langle t_{\delta(i,j)} \rangle & \delta(i,j) \in S^{\mathrm{univ}} \\ a_{ij} & 1 \le i \le d, j = h \end{cases}$$

While the coordinates t_{ij} on the deformation space arise naturally from our choice of normal display, they obscure the Newton stratification of that deformation space. We introduce a new set of coordinates $\widetilde{u}_{x,y}$ on R^{univ} adapted to Newton polygon calculations, as follows.

Let $\mathcal{P} = f(\delta^{-1}(S^{\mathrm{univ}}))$; it is the set of lattice points in the parallelogram with vertices $(1, 0)$, $(d, 0)$, $(c, c-1)$ and $(h-1, c-1)$ (Figure 1). For $(i, j) \in S^{\mathrm{univ}}$, let $\widetilde{u}_{x(\delta^{-1}(i,j)),y(\delta^{-1}(i,j))} = t_{i,j}$. Then there is a canonical isomorphism

$$R^{\mathrm{univ}} = k[\![\widetilde{u}_{x,y} : (x, y) \in \mathcal{P}]\!], \tag{3.9}$$

and the characteristic polynomial of M^{univ} is

$$\chi_{\mathrm{univ}}(F) = \chi_0(F) + \sum_{(x,y)\in\mathcal{P}} p^y \langle \widetilde{u}_{x,y} \rangle^{\sigma^{h-d-y}} F^{h-x}.$$

Note that this gives a formula for computing the characteristic polynomial of Frobenius of any deformation of M_0. Indeed, let $\phi : R^{\mathrm{univ}} \to R$ be a map to a complete local ring, necessarily of positive characteristic. Write $r_{x,y} = \phi(\widetilde{u}_{x,y})$; then the characteristic polynomial of Frobenius of the deformation M/R of M_0 is

$$\chi(F) = \chi_0(F) + \sum_{(x,y)\in\mathcal{P}} p^y \langle r_{x,y} \rangle^{\sigma^{h-d-y}} F^{h-x}.$$

3.4. **Construction of a deformation.** We continue to work with a Dieudonné module M_0 with $a(M_0) = 1$ and a positive slope λ smaller than any slope of M_0 but still attainable from M_0. The possibilities for λ are completely determined by Theorem 2.1. Write $\lambda = r/s$ with $\gcd(r,s) = 1$. For λ to be attainable, it is necessary and sufficient that $r < c$, and that the slope of the line segment from (s,r) to (h,c) satisfies $\lambda \leq (c-r)/(h-s) \leq 1$. In particular, we may and do assume that $s > r$ and that $s \leq r + d$.

We recapitulate the method of Oort for constructing deformations, and then obtain details about the characteristic polynomial of the resulting deformed module. As in the beginning of this section, let $\mathrm{NP}(*)$ denote the Newton polygon obtained by adjoining the point (s,r) to the Newton polygon of M_0; that is, $\mathrm{NP}(*)$ is the convex hull of $\mathrm{NP}(M_0) \cup \{(s,r)\}$. Since the Newton polygon of a Dieudonné module with a-number one is the same as the Newton polygon of its characteristic polynomial, we can control the Newton polygon of a deformation of M_0 by examining the p-adic ordinals of the coefficients of its characteristic polynomial.

Define

$$\begin{aligned}\mathcal{P}(*) &:= \{(x,y) \in \mathcal{P} : (x,y) \text{ lies on or above } \mathrm{NP}(*)\}\\ R &:= k[\![u_{xy} : (x,y) \in \mathcal{P}(*)]\!].\end{aligned}$$

We define a deformation of M_0/k to M/R by specializing the universal deformation:

$$R^{\mathrm{univ}} \xrightarrow{\ \phi\ } R \tag{3.10}$$

$$\widetilde{u}_{xy} \longmapsto \begin{cases} u_{xy} & (x,y) \in \mathcal{P}(*) \\ 0 & (x,y) \notin \mathcal{P}(*) \end{cases}.$$

By [10, 2.6], the Newton polygon of M is indeed $\mathrm{NP}(*)$. We can say more. Any deformation of G_0 to a complete local domain such that the Newton polygon of the deformed p-divisible group lies on or above $\mathrm{NP}(*)$ arises from this deformation. This, together with the fact that R is a domain, characterizes this deformation.

Set $a_x(M_0) = p^{-\lambda x} A_x(M_0)$, and $a_x(M) = p^{-\lambda x} A_x(M)$. By Lemma 3.2, the monodromy group of M in slope λ is the Galois group generated by the Witt components of any v which satisfies

$$v^{\sigma^h} - \sum_{x=0}^{h-1} \left(a_x(M_0) + \sum_{y:(x,y)\in\mathcal{P}(*)} p^{y-\lambda x} \langle u_{x,y} \rangle^{\sigma^{h-d-y}} \right) v^{\sigma^{h-x}} = 0. \tag{3.11}$$

To prove that this Galois group is large, we need control over some (in fact, three) of the terms which appear in (3.11). For any nonnegative integer j, let

$$\begin{aligned}\mathcal{P}(j) &= \{(x,y) : (x,y) \in \mathcal{P}(*), y - \lambda x = j/s\}\\ \overline{\mathcal{P}}(j) &= \cup_{i \leq j} \mathcal{P}(j).\end{aligned}$$

We will see in Section 3.5 that partitioning $\mathcal{P}(*)$ as $\cup \mathcal{P}(j)$ corresponds to keeping track of terms with p-adic valuation j/s. In the present section, our goal is to show that $\mathcal{P}(0)$, $\mathcal{P}(1)$ and $\mathcal{P}(s)$ are nonempty.

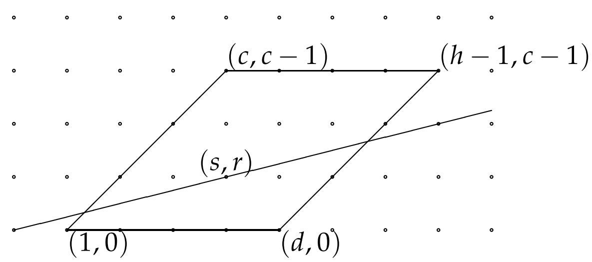

FIGURE 1. *The parallelogram $\mathcal{P}$.*

Lemma 3.6. $\mathcal{P}(0) = \{(s, r)\}$.

Proof. Certainly, (s, r) is in $\mathcal{P}(0)$. Let (t, u) be any lattice point with $t \geq 0$ and $\lambda u = t$. If $t < s$, then $\gcd(r, s) \neq 1$, contradicting our original hypothesis on the representation $\lambda = r/s$. If $t > s$ and $(t, u) \in \mathcal{P}$, then the sub-Dieudonné module with slope at most λ would have height greater than s, which contradicts the definition of NP$(*)$. □

Lemma 3.7. *If $(s, r) \neq (s, s-1)$, then $\mathcal{P}(1)$ is nonempty.*

Proof. Write $(s, r) = (a + b, a)$ with $\gcd(a, b) = 1$, and consider Figure 1. Since the segment from $(a + b, a)$ to $(c + d, c)$ has slope at most one, it follows that $b \leq d$. Since the Newton polygon is lower convex we may assume that $a \leq c - 1$.

By hypothesis $b > 1$. The line $y = (a/(a + b))x$ enters the parallelogram $\mathcal{P}$ at $x = (a + b)/b$ and continues inside the parallelogram at least until the point $(a + b, a)$ since $d \geq b$. For any integer α between $(a + b)/b$ and $a + b$, the point with integer coordinates directly on or above $(\alpha, \alpha\lambda) = (\alpha, (a\alpha)/(a + b))$ is in $\mathcal{P}$. Thus it suffices to show that there is an α in this range so that

$$a\alpha/(a + b) \equiv -1/(a + b) \mod \mathbb{Z}$$

or

$$a\alpha \equiv -1 \mod a + b,$$

which is equivalent to

$$b\alpha \equiv 1 \mod a + b.$$

There is a solution to this equation with α in the range $1 \leq \alpha < a + b$, since $\gcd(a, b) = 1$. There is no solution in the range $1 \leq \alpha \leq (a + b)/b$; hence there is a solution in the range $(a + b)/b < \alpha < a + b$. □

Lemma 3.8. *If $(s, r) \neq (s, s-1)$, then $\mathcal{P}(s)$ is nonempty.*

Proof. Since $r \leq s - 2$, $(s, r + 1) \in \mathcal{P}(*)$. □

3.5. **Calculation of Galois action.** Write $a_x = \sum p^{j/s}\langle a_{x,j}\rangle$ with $a_{x,j} \in k$. By using Lemma 3.6 and recalling that $a_{x,0} = 0$, we can write our monodromy equation (3.11) as

(3.12)
$$v^{\sigma^h} - \langle u_{s,r}\rangle^{\sigma^{h-d-r}} v^{\sigma^{h-s}} - \sum_{j\geq 1}\sum_x p^{j/s} v^{\sigma^{h-x}} \left(\langle a_{x,j}\rangle + \sum_{y:(x,y)\in\mathcal{P}(j)} \langle u_{(x,y)}\rangle^{\sigma^{h-d-y}} \right) = 0.$$

(For given x and j, if there is no y such that $(x, y) \in \mathcal{P}(j)$ then we set the innermost sum equal to zero.) Write $v = \sum p^{j/s}\langle v_j\rangle$, $K = \operatorname{Frac} R$, and $K_j = K(v_0, \cdots, v_j)$. For a subset $P \subset \mathcal{P}(*)$, let $k[\![u_P]\!] = k[\![u_{(x,y)} : (x, y) \in P]\!]$, with field of fractions $k(\!(u_P)\!)$.

Lemma 3.9. $\operatorname{Gal}(K_0/K) \cong \mathbb{F}_q^\times$.

Proof. Reduce equation (3.12) modulo $p^{1/s}$ and consider the initial Witt component of v: it satisfies

$$v_0^{p^h} - u_{r,s}^{p^{h-d-r}} v_0^{p^{h-s}} = 0. \tag{3.13}$$

Let t be a nonzero solution to (3.13). It satisfies

$$t^{p^h - p^{h-s}} = u_{r,s}^{p^{h-d-r}}, \tag{3.14}$$

and generates an extension of separable degree $p^s - 1$ over $k(\!(u_{\mathcal{P}(*)})\!)$ whose Galois group is $\mathbb{F}_q^\times$. □

We continue to let $t = v_0$. The equality (3.14) in the algebraic closure of $k(\!(u_{\mathcal{P}})\!)$ implies an equality of Witt vectors

$$\frac{\langle u_{s,r}\rangle^{\sigma^{h-d-r}}}{\langle t\rangle^{\sigma^h - \sigma^{h-s}}} = 1. \tag{3.15}$$

In (3.12) set $v = \langle t\rangle w$ and divide by $\langle t\rangle^{\sigma^h}$. Using (3.15), we obtain

$$w^{\sigma^h} - w^{\sigma^{h-s}} - \langle t\rangle^{-\sigma^h} \sum_{j\geq 1}\sum_x p^{j/s}\langle t\rangle^{\sigma^{h-x}} w^{\sigma^{h-x}} \left(a_{x,j} + \sum_{y:(x,y)\in\mathcal{P}(j)} \langle u_{(x,y)}\rangle^{\sigma^{h-d-y}} \right) = 0.$$

Write $w = \sum_{i=0} p^{i/s}\langle w_i\rangle$ with $w_0 = 1$. Then our equation becomes

$$\begin{aligned}
&(\sum_i p^{i/s}\langle w_i\rangle)^{\sigma^h} - (\sum_i p^{i/s}\langle w_i\rangle)^{\sigma^{h-s}} \\
&- \langle t\rangle^{-\sigma^h} \sum_{j\geq 1}\sum_x p^{j/s} \left(\langle a_{x,j}\rangle + \sum_{y:(x,y)\in\mathcal{P}(j)} \langle u_{(x,y)}\rangle^{\sigma^{h-d-y}} \right) \left(\langle t\rangle^{\sigma^{h-x}} \sum_{i\geq 0} p^{i/s}\langle w_i\rangle^{\sigma^{h-x}} \right) = 0
\end{aligned}$$

or, regrouping,

$$\begin{aligned}
&(\sum_\ell p^{\ell/s}\langle w_\ell\rangle)^{\sigma^h} - (\sum_\ell p^{\ell/s}\langle w_\ell\rangle)^{\sigma^{h-s}} \\
&- \langle t\rangle^{-\sigma^h} \sum_{\ell\geq 1} p^{\ell/s} \sum_{j=1}^{\ell}\sum_x \left(\langle a_{x,j}\rangle + \sum_{y:(x,y)\in\mathcal{P}(j)} \langle u_{x,y}\rangle^{\sigma^{h-d-y}} \right) \left(\langle t\rangle^{\sigma^{h-x}} \langle w_{\ell-j}\rangle^{\sigma^{h-x}} \right) = 0.
\end{aligned}$$

Inductively, for $1 \leq \ell \leq s$ we have

$$w_\ell^{p^h} - w_\ell^{p^{h-s}} - t^{-p^h} \sum_{j=1}^{\ell} \sum_x \left(a_{x,j} + \sum_{y:(x,y)\in\mathcal{P}(j)} u_{(x,y)}^{\sigma^{h-d-y}} \right) (t^{p^{h-x}} w_{\ell-j}^{p^{h-x}}) = 0. \tag{3.16}$$

Since we have equality of Witt vectors in equation (3.15) the lifts $\langle w_j \rangle$ for $1 \leq j < \ell$ solve the Witt equation modulo $p^{(1+s)/s}$. The higher-order terms are irrelevant in our calculations since $\ell \leq s$. Moreover, w_ℓ is defined in terms of $w_0, \cdots, w_{\ell-1}$ and coordinates $u_{x,y}$ for $(x,y) \in \overline{\mathcal{P}}(\ell)$, so that w_ℓ is algebraic over $k((u_{\overline{\mathcal{P}}(\ell)}))$.

Let $\mathcal{Q}(\ell) = \overline{\mathcal{P}}(\ell) - \{(s,r)\}$. With this notation, $w_1, \cdots, w_{\ell-1}$ are algebraic over $k((u_{\mathcal{Q}(\ell)}))((t))$.

Consider the extension $k((u_{\mathcal{Q}(\ell-1)}))((t)) \subset k((u_{\mathcal{Q}(\ell-1)}))((t))(w_1, \cdots w_{\ell-1})$. Since $k((u_{\mathcal{Q}(\ell-1)}))[\![t]\!]$ is a discrete valuation ring, we can write this extension as

$$k((u_{\mathcal{Q}(\ell-1)}))((t)) \subset L_{\ell-1}((t)) \subset L_{\ell-1}((t_{\ell-1})),$$

where $L_{\ell-1}$ is an extension of the residue field $k((u_{\mathcal{Q}(\ell-1)}))$, and the second extension is totally ramified with uniformizing parameter $t_{\ell-1}$.

The equation for w_ℓ is defined over

$$k((u_{\overline{\mathcal{P}}(\ell)}))(w_\ell, \cdots w_{\ell-1}) \subset L_{\ell-1}((u_{\mathcal{P}(\ell)}))((t_{\ell-1})),$$

and has the form

$$w_\ell^{p^h} - w_\ell^{p^{h-s}} - t^{-p^h}\Big(\sum_{(x,y)\in\mathcal{P}(\ell)} u_{x,y}^{p^{h-d-y}} t^{p^{h-x}} \Big) - B = 0$$

with $B \in L_{\ell-1}((t_{\ell-1}))$. Write $A = t^{-p^h}(\sum_{(x,y)\in\mathcal{P}(\ell)} u_{x,y}^{p^{h-d-y}} t^{p^{h-x}})$.

Lemma 3.10. *Assume $1 \leq \ell \leq s$. If there exists some $(x_0, y_0) \in \mathcal{P}(\ell)$, then the separable degree of the extension $L_\ell/L_{\ell-1}$ is greater than or equal to p^s.*

Proof. Let $z = w^{p^{h-s}}$, then z is a root of $X^{p^s} - X = A + B$. This equation is separable; we show it is irreducible. By Lemma A.1, it suffices to show that it is impossible to write $A + B = f_H(x)$, where H is a nontrivial subgroup of $\mathbb{F}_{p^s}$, $f_H(x) = \prod_{a\in H}(x-a)$, and $x \in L_{\ell-1}((u_{\mathcal{P}(\ell)}))((t_{\ell-1}))$.

To show that $f_H(X) = A + B$ has no solution we use Lemma A.2. Note that the field k in the appendix corresponds to the field $L_{\ell-1}$ here; the variable t in the appendix corresponds to $t_{\ell-1}$ here; and the variables $z_1, \cdots z_e$ in the appendix correspond to $u_{x,y}$ with $(x,y) \in \mathcal{P}(\ell)$. □

Lemma 3.11. *Suppose $r \neq s-1$. Then $\mathrm{Gal}(K_0/K) \cong \mathbb{F}_q^\times$, while $[K_1 : K_0] \geq p^s$ and $[K_s : K_{s-1}] \geq p^s$.*

Proof. The first claim is Lemma 3.9, while the rest follows immediately from Lemmas 3.7, 3.8 and 3.10. □

The main result of this section now follows easily.

Proof of Lemma 3.1. We show that the deformation defined by (3.10) has large monodromy in slope λ. The monodromy group admits a filtration by subgroups of the form $1 + \varpi^n \mathcal{O}_\lambda$. By comparing Lemma 3.11 with the cardinality of the graded pieces of $\mathcal{G}_\lambda$ (3.3), the monodromy group is maximal for graded pieces 0, 1 and s. By Lemma 3.5, we conclude that the λ-monodromy group is all of $\mathrm{Aut}(N_{\lambda,F})$. □

4. MAIN RESULT

The goal of this section is to complete the proof of

Theorem 4.1. *Let G_0/k be a p-divisible group. Let λ be a rational number which is not a slope of G_0 but is attainable from G_0. Assume that $\lambda \neq (s-1)/s$ for any natural number $s \geq 2$. Then there exists a smooth equicharacteristic deformation G_R of G_0/k to a domain R so that G_R attains λ, and the monodromy group of G_K in slope λ is large.*

The theorem follows from Lemma 3.1 and the three lemmas below. Lemma 4.2 removes the hypothesis on $a(G_0)$ from Lemma 3.1. Lemma 4.3 completes the proof of the main theorem for positive λ by reducing to the case where λ is strictly less than any slope of G_0. Lemma 4.5 proves the theorem in case λ=0.

Lemma 4.2. *Let G_0 be a local p-divisible group over an algebraically closed field k. Let $\lambda \in [0,1], \lambda \neq (s-1)/s$, be a positive rational number attainable from G_0 which is strictly less than any slope of G_0. Then there exists a deformation of G_0 to a p-divisible group G which attains λ with multiplicity one and that has large λ-monodromy.*

Proof. Suppose that M_0 has dimension d and codimension c. Choose a display for M_0 as in (3.4); then the universal deformation of M_0, defined over R^{univ} and denoted M^{univ}, is given in (3.8).

Oort shows [11] that there exists a complete local domain R and elements $\overline{r_{ij}} \in R$ so that, if we write r for the matrix $(\langle\overline{r_{ij}}\rangle)$, then the Dieudonné module M_1 displayed by

$$\begin{pmatrix} A+rC & B+rD \\ C & D \end{pmatrix}$$

has the same Newton polygon as M_0, but $a(M_1(\operatorname{Frac} R)) = 1$.

Let $S_{ij} = \langle s_{ij}\rangle \in W(R[\![s_{ij} : (i,j) \in S^{\mathrm{univ}}]\!])$. Let $S = (S_{ij})$, and let M_2 be the Dieudonné module over $R[\![s_{ij}]\!]$ with display

$$\begin{pmatrix} A+rC+SC & B+rD+SD \\ C & D \end{pmatrix}. \tag{4.1}$$

The Dieudonné module M_2 is a deformation of M_0. Indeed, the map

$$R^{\mathrm{univ}} \xrightarrow{\phi_1} R[\![s_{ij}]\!]$$

$$t_{ij} \longmapsto r_{ij} + s_{ij}$$

exhibits M_2 as a deformation of M_0.

Let K be the fraction field of R and $\overline{K}$ its algebraic closure. The Dieudonné module $M_2(K[\![s_{ij}]\!])$ defined by (4.1) is visibly the universal deformation of $M_1(K)$.

Let λ be a slope attainable from M_0 which is strictly less than any slope of M_0. Write $\lambda = r/s$ where r and s are relatively prime integers. Let $\mathrm{NP}(*)$ denote the lower convex hull of the Newton polygon of M_0 with (s,r) adjoined. By hypothesis, $\mathrm{NP}(*)$ has the same beginning and end points as the Newton polygon of M_0.

By a result of Katz [8, 2.3.1], the locus of $\operatorname{Spf} R^{\mathrm{univ}}$ over which the Newton polygon of the deformation lies on or above $\mathrm{NP}(*)$ is Zariski-closed. Indeed, he shows there exists an ideal $J \subset R^{\mathrm{univ}}$ so that, if $\phi : R^{\mathrm{univ}} \to R'$ is any extension

of scalars to a domain R' inducing a Dieudonné module, then $\phi(J)R'$ defines (set-theoretically) the locus of points where the Newton polygon $M^{\mathrm{univ}}(R')$ lies on or above $\mathrm{NP}(*)$.

We apply this to our situation. Consider the sequence of homomorphisms

$$R^{\mathrm{univ}} \longrightarrow R[\![s_{ij}]\!] \longrightarrow K[\![s_{ij}]\!] \longrightarrow \overline{K}[\![s_{ij}]\!].$$

The ideals J, $JR[\![s_{ij}]\!]$, $JK[\![s_{ij}]\!]$, and $J\overline{K}[\![s_{ij}]\!]$ define the loci where M^{univ}, M_2, $M_2(K)$, and $M_2(\overline{K})$, respectively, specialize to a Dieudonné module with Newton polygon on or above $\mathrm{NP}(*)$.

Let $Y = R[\![s_{ij}]\!]/JR[\![s_{ij}]\!]$. Then $M_2(Y)$ is a deformation of M_0 whose Newton polygon is equal to $\mathrm{NP}(*)$. Our goal is to show that there is a point of $\mathrm{Spf}(Y)$ so that $M_2(Y)$ restricted to that point has generic Newton polygon equal to $\mathrm{NP}(*)$ and has large monodromy.

Let $Y_{\overline{K}}$ denote $Y \otimes_R \overline{K}$. The a-number of $M_1(\overline{K})$ is one. Since $M_2(\overline{K}[\![s_{ij}]\!])$ is the universal deformation of $M_1(\overline{K})$, and since $J\overline{K}[\![s_{ij}]\!]$ defines the locus of deformations of $M_1(\overline{K})$ with Newton polygon on or above $\mathrm{NP}(*)$, we can apply Lemma 3.1 to conclude that the Dieudonné module $M_2(Y_{\overline{K}})$ has large monodromy.

Let I be the kernel of the map $Y \to Y_K$ and set $Y' = Y/I$. We will show that $M_2(Y')$ has large monodromy. To see this let $Q_{\overline{K}}$ be the field of fractions of $Y_{\overline{K}}$, and let Q' be the field of fractions of Y'.

$$\begin{tikzcd}
 & & Q_{\overline{K}} = \operatorname{Frac} Y_{\overline{K}} \arrow[dll, dash] \arrow[d, dash] \\
Y_{\overline{K}} \arrow[d, dash] & & Q' = \operatorname{Frac} Y' \subset \operatorname{Frac} Y_K \arrow[dll, dash] \\
Y' \subset Y_K & &
\end{tikzcd}$$

Let N_λ be the slope-λ test object; it is defined over $\mathbb{F}_p$. We can replace the p-divisible group associated to M_2 over the field Q' by its local part. Hence we proceed assuming that M_2 is local. Let L be any algebraic extension of Q' such that there exists a nontrivial map

$$N_\lambda \xrightarrow{\phi} M_2(Q'.L).$$

We need to show that the action of $\mathrm{Gal}(L/Q')$ is large.

Now, ϕ induces a map $\phi_{\overline{K}}$:

$$N_\lambda \xrightarrow{\phi_{\overline{K}}} M_2(Q_{\overline{K}}.L).$$

By Lemma 3.1, the action of $\mathrm{Gal}(Q_{\overline{K}}.L/Q_{\overline{K}})$ is large. The diagram of fields

$$\begin{tikzcd}
 & Q_{\overline{K}}.L \arrow[dl, dash] \arrow[d, dash] \\
Q_{\overline{K}} \arrow[d, dash] & L \arrow[dl, dash] \\
Q' &
\end{tikzcd}$$

yields an inclusion $\mathrm{Gal}(Q_{\overline{K}}.L/Q_{\overline{K}}) \hookrightarrow \mathrm{Gal}(L/Q')$. Therefore, $M_2(Y')$ has large λ-monodromy. □

Lemma 4.3. *Let G_0 be a p-divisible group over an algebraically closed field k. Let $\lambda \in [0,1]$ be a positive rational number attainable from G_0 which is not a slope of G_0. Then there exists a deformation of G_0 to a complete local domain with residue field k that has large λ-monodromy.*

Proof. Since a p-divisible group over a field always admits a slope filtration (2.1), there exists a filtration

$$(0) = G_0^{(0)} \subseteq G_0^{(1)} \subseteq G_0^{(2)} \subseteq G_0^{(3)} = G_0$$

such that:

- For $1 \le i \le 3$, the subquotient $H_0^{(i)} := G_0^{(i)}/G_0^{(i-1)}$ is a p-divisible group.
- The slopes of $G_0^{(1)}$ are all less than λ; the slopes of $H_0^{(2)}$ are all greater than λ; the slopes of $G_0^{(2)}$ are all less than 1; and $H_0^{(3)}$ is étale.

By definition of attainability λ is attainable from G_0 if and only if it is attainable from $G_0/G_0^{(1)}$. Moreover, λ is attainable from $G_0/G_0^{(1)}$ if and only if it is attainable from $H_0^{(2)}$; if not, a slope strictly larger than 1 would appear in a p-divisible group which attained λ. By Sublemma 4.4, a deformation $H^{(2)}/R$ of $H_0^{(2)}$ lifts to a deformation G/R of G_0 as filtered p-divisible group. Moreover, $\mathrm{Hom}_{\overline{K}}(H_\lambda, G_{\overline{K}}) = \mathrm{Hom}_{\overline{K}}(H_\lambda, H_{\overline{K}}^{(2)})$. Therefore, there exists a deformation of G_0 with large monodromy in slope λ if there exists such a deformation of $H^{(2)}$. □

Sublemma 4.4. *Let $(0) = G_0^{(0)} \subseteq G_0^{(1)} \subseteq \cdots \subseteq G_0^{(r)} = G_0$ be a filtered p-divisible group over k whose quotients $H_0^{(i)} := G_0^{(i)}/G_0^{(i-1)}$ are p-divisible groups such that $H_0^{(r)}$ is étale and $G_0^{(r-1)}$ is local. Let R be a local k-algebra, and for each i let $H^{(i)}$ be a deformation of $H_0^{(i)}$ to R. Then there exists a deformation $G^{(0)} \subseteq G^{(1)} \subseteq \cdots \subseteq G^{(r)}$ of G_0 as filtered p-divisible group over R such that $G^{(i)}/G^{(i-1)} \cong H^{(i)}$ for each i.*

Proof. This is essentially contained in [11, 2.4].

First, we reduce to the case where the étale part of G_0 is trivial. Indeed, suppose $G_0^{(1)} \subset G_0^{(2)} = G_0$ is an inclusion of p-divisible groups so that $G_0^{(1)}$ is local and $G_0^{(2)}/G_0^{(1)}$ is étale. Then G_0 admits a decomposition as a direct sum $G_0 \cong (G_0/G_0^{(1)}) \oplus G_0^{(1)} = H_0^{(2)} \oplus H_0^{(1)}$. If $H^{(i)}$ is a deformation of $H_0^{(i)}$ over R for $i = 1, 2$, then $H^{(1)} \subset H^{(1)} \oplus H^{(2)}$ is a suitable deformation of $G_0^{(1)} \subset G_0^{(2)}$.

Henceforth, we assume that G_0 is local, so that $G_0^{(r-1)} = G_0^{(r)}$. Denote the dimension and codimension of $G_0^{(i)}$ by $d(i)$ and $c(i)$, respectively. There is a $W(k)$-basis $x_1, \cdots, x_{d(r)}, y_1, \cdots, y_{c(r)}$ for $M = \mathbb{D}_*(G)$ so that $x_1, \cdots, x_{d(i)}, y_1, \cdots, y_{c(i)}$ is a $W(k)$-basis for $\mathbb{D}_*(G_0^{(i)})$. We describe the display $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to this basis.

The matrix A is block-upper-triangular, with blocks $A_1, \cdots A_r$. The matrix A_i is square of size $d(i) - d(i-1)$; B, C and D have an analogous structure. Observe that $\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ is a display for $H_0^{(i)}$.

Let R be any local k-algebra, and suppose that a deformation $H^{(i)}/R$ of $H^{(i)}/k$ is given for each i. Such a deformation is described by a display $\begin{pmatrix} A_i + T^{(i)}C_i & B_i + T^{(i)}D_i \\ C_i & D_i \end{pmatrix}$, where $T^{(i)}$ is a $(d(i) - d(i-1)) \times (c(i) - c(i-1))$ matrix with entries in R.

Let T be the block-diagonal matrix with blocks $T^{(1)}, \cdots, T^{(r)}$, and use this to construct the display $\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}$ over R. This defines a p-divisible group G/R. Since each of $A + TC$, $B + TD$, C and D is block-upper-triangular, the filtration of G_0 extends to a filtration $G^{(0)} = 0 \subset G^{(1)} \subset \cdots \subset G^{(r)} = G$ over R. Moreover, the display of $H^{(i)}$ is identical to that of $G^{(i)}/G^{(i-1)}$, so that these are isomorphic p-divisible groups. □

We conclude this section by proving Theorem 4.1 for $\lambda = 0$. Note that this gives a (somewhat anachronistic) proof of Igusa's result [7]. We include the analogous result for slope one, even though such a slope is not "attainable" in the formulation of Definition 1.1.

Lemma 4.5. *Let R be a complete local ring with field of fractions K and residue field k. Let G/R be a p-divisible group, with special and generic fibers G_0 and G_K, respectively.*

(a) *Suppose the multiplicities of slope zero in G_K and G_0 are 1 and 0, respectively. Then the slope zero monodromy of G_K has finite index in $\mathbb{Z}_p^\times$.*
(b) *Suppose the multiplicities of slope one in G_K and G_0 are 1 and 0, respectively. Then the slope one monodromy of G_K has finite index in $\mathbb{Z}_p^\times$.*

Proof. For (a), consider the representation

$$\mathrm{Gal}(K) \longrightarrow \mathrm{Aut}(\mathrm{Hom}_{\overline{K}}(\boldsymbol{\mu}_{p^\infty}, G_{\overline{K}})) \cong \mathbb{Z}_p^\times.$$

Since the representation is continuous, the image of this map is closed. In fact, this image is actually infinite. If not, then over some finite extension K'/K there would exist a nontrivial homomorphism from $\boldsymbol{\mu}_{p^\infty,K'}$ to $G_{K'}$, which would necessarily extend [2, 1.2] to the integral closure of R in K'. This contradicts the hypothesis that 0 is not a slope of G_0. In particular, the monodromy group contains some non-torsion element α. Let $\beta = \alpha^{p-1}$. Then there is some $n \geq 1$ so that $\beta \equiv 1 \bmod p^n$ but $\beta \not\equiv 1 \bmod p^{n+1}$. Therefore, β generates the group $(1 + p^n\mathbb{Z}_p)/(1 + p^{n+1}\mathbb{Z}_p)$. Since the Frattini subgroup of $1 + p^n\mathbb{Z}_p$ is $1 + p^{n+1}\mathbb{Z}_p$, β topologically generates $1 + p^n\mathbb{Z}_p$, a subgroup of finite index in $\mathbb{Z}_p^\times$.

The proof of (b) is similar. There is a subgroup $G_K^{(1)} \subset G_K$ such that $G_K/G_K^{(1)}$ is geometrically isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, and the monodromy of G in slope one is the same as that of $G_K/G_K^{(1)}$. We claim that this monodromy group is infinite; as in part (a), this implies that the monodromy group has finite index in $\mathbb{Z}_p^\times$.

Indeed, suppose not. Over some finite extension K'/K there would exist a nontrivial homomorphism $G_K \to \mathbb{Q}_p/\mathbb{Z}_p$. By [2, 1.2], this would extend to a nontrivial homomorphism $G_{R'} \to \mathbb{Q}_p/\mathbb{Z}_p$, where R' is the integral closure of R in K'. This contradicts the hypothesis that G_0 is purely local. $\square$

5. Polarizations

We now explain how to extend Theorem 4.1 to the setting of principally quasi-polarized p-divisible groups, since these are the ones which arise in applications to abelian varieties. For such a p-divisible group the dimension and codimension are the same, and we set $c = d = g$, $h = 2g$.

Recall that by a principally quasi-polarized (pqp) p-divisible group we mean a p-divisible group G equipped with a self-dual isomorphism $\Phi : G \to G^t$. In this section, deformations will always be of pqp p-divisible groups. Recall that, given a deformation of a p-divisible group, if a chosen quasi-polarization also deforms then it does so uniquely.

Note that a slope λ appears in the Newton polygon of a pqp p-divisible group with same multiplicity as $1-\lambda$. Moreover a pqp p-divisible group has large monodromy in slope λ if and only if it has large monodromy in slope $1-\lambda$.

Theorem 5.1. *Let G_0 be a principally quasi-polarized p-divisible group over an algebraically closed field. Suppose that λ is not a slope of G_0, that $\lambda \neq (s-1)/s$ for any natural number $s \geq 2$, and that λ is symmetrically attainable from G_0. Then there exists a pqp deformation of G_0 which symmetrically attains λ with large λ-monodromy.*

Proof. Since the proof is similar to that for p-divisible groups without polarization, we limit ourselves to the points where the proofs differ. Observe that slope $1/2$ is not symmetrically attainable. Since a slope λ occurs in a pqp p-divisible group if and only if $1-\lambda$ occurs, it suffices to consider the case $\lambda < 1/2$. The proof of Lemma 4.5 holds in the presence of a principal quasi-polarization, so we assume that $\lambda > 0$.

First, we show that it suffices to prove the theorem under the additional hypothesis that λ is strictly less than any slope of G_0. We use techniques from Section 3 of Oort's article [11], in particular Lemmas 3.5, 3.7 and 3.8 there. He proves that there exists a filtration by p-divisible groups

$$(0) = G_0^{(0)} \subseteq G_0^{(1)} \subseteq G_0^{(2)} \subseteq G_0^{(3)} = G_0$$

such that:

- For $1 \leq i \leq 3$, the subquotient $H_0^{(i)} := G_0^{(i)}/G_0^{(i-1)}$ is a p-divisible group.
- The slopes of $G_0^{(1)}$ are all less than λ; the slopes of $H_0^{(3)}$ are all greater than $1-\lambda$; the slopes of $G_0^{(2)}$ are all less than $1-\lambda$.
- The filtration is symplectic, in the sense that the principal quasi-polarization $\Phi_0 : G_0 \to (G_0)^t$ induces isomorphisms $\Phi_0^{(i)} : H_0^{(i)} \to (H_0^{(3-i+1)})^t$.

In particular, the pair $(H_0^{(2)}, \Phi_0^{(2)})$ is a pqp p-divisible group.

We decompose the Newton polygon of G_0 into three parts. The initial, middle and final parts respectively arise from $H_0^{(1)}$, $H_0^{(2)}$ and $H_0^{(3)}$. Oort shows that a deformation of the pqp p-divisible group $(H_0^{(2)}, \Phi^{(2)})$ to a ring R lifts to a deformation (G, Φ) of (G_0, Φ_0) so that the inclusion $G_0^{(1)} \hookrightarrow G_0$ extends to $G_0^{(1)} \times R \hookrightarrow G$. In particular, the Newton polygon of G has the same initial and final parts as G_0. Thus, to prove the theorem it suffices to show that there exists a deformation of $(H_0^{(2)}, \Phi_0^{(2)})$ to a pqp p-divisible group whose Newton polygon is the same as that of the middle part of G and which has large λ-monodromy. In particular, it suffices to prove the theorem under the hypothesis that λ is smaller than any slope of G_0.

Second, we show that we may assume that $a(G_0) = 1$. By [11, Corollary 3.10], given a pqp p-divisible group over a field, there exists a pqp deformation with the same Newton polygon as the original group but with generic a-number one. The reduction argument of Lemma 4.2 now applies.

We now assume that $a(G_0) = 1$, that $\lambda > 0$, and that λ is smaller than all slopes of G_0. Since G_0 has no toric part and is self-dual, it is local; we can use covariant Dieudonné theory. Let $M_0 = \mathbb{D}_*(G_0)$, and let g denote the dimension and codimension of M_0. By a result of Oort [10, 2.3] we can find a $W(k)$-basis $\{e_1, \cdots, e_{2g}\}$ for M_0 so that the display of M_0 is normal and the pairing takes the form

$$\langle e_i, e_j \rangle = \begin{cases} 1 & j = i+g \\ -1 & j = i-g \\ 0 & |i-j| \neq g \end{cases} .$$

Define an involution on S^{univ} by $\mathrm{inv}_M((i,j)) = (j-g, i+g)$. The universal pqp deformation $M^{\mathrm{univ,pol}}$ of M_0 is defined over the ring

$$R^{\mathrm{univ,pol}} := R^{\mathrm{univ}}/(t_{ij} - t_{\mathrm{inv}_M((i,j))} : (i,j) \in S^{\mathrm{univ}}).$$

We calculate the equation for $R^{\mathrm{univ,pol}}$ with respect to the new coordinates on R^{univ} introduced in (3.9). Let $\mathrm{inv}_{\mathrm{NP}}(x,y) = (2g-x, g-x+y)$; we then have

$$R^{\mathrm{univ,pol}} = R^{\mathrm{univ}}/(\widetilde{u}_{x,y} - \widetilde{u}_{\mathrm{inv}_{\mathrm{NP}}(x,y)}).$$

Let $\mathrm{NP}(*)_{\mathrm{pol}}$ denote the symmetric Newton polygon with endpoints $(0,0)$ and $(2g, g)$ obtained from the Newton polygon of M_0 by adjoining (s,r). It is the lower convex hull of $\mathrm{NP}(M_0) \cup \{(s,r), \mathrm{inv}_{\mathrm{NP}}(s,r)\}$. If $0 \le x \le s$, note that $(x,y) \in \mathrm{NP}(*)_{\mathrm{pol}}$ if and only if $\mathrm{inv}_{\mathrm{NP}}(x,y) \in \mathrm{NP}(*)_{\mathrm{pol}}$.

The universal pqp deformation of $(M_0, \langle \cdot, \cdot \rangle)$ with Newton polygon on or above $\mathrm{NP}(*)_{\mathrm{pol}}$ is defined over $R^{\mathrm{pol}} := R \otimes_{R^{\mathrm{univ}}} R^{\mathrm{univ,pol}}$, where R is the ring constructed in Section 3.4; the associated p-divisible group is the pullback of the tautological group $G^{\mathrm{univ}}/R^{\mathrm{univ}}$.

Furthermore, the analysis of section three goes through for G/R^{pol}, too. Specifically, let $\mathcal{P}(*)_{\mathrm{pol}} = \{(x,y) \in \mathcal{P} : (x,y) \text{ lies on or above } \mathrm{NP}(*)_{\mathrm{pol}}\}$, and let $\mathcal{P}(j)_{\mathrm{pol}} = \mathcal{P}(j) \cap \mathcal{P}(*)_{\mathrm{pol}}$. Then Lemmas 3.6, 3.7 and 3.8 apply to $\mathcal{P}(*)_{\mathrm{pol}}$, too, and we have $\mathcal{P}(0)_{\mathrm{pol}} = \{(s,r)\}$, and $\mathcal{P}(1)_{\mathrm{pol}}$ and $\mathcal{P}(s)_{\mathrm{pol}}$ are nonempty. Therefore (cf. Section 3.5) $G^{\mathrm{pol}}/R^{\mathrm{pol}}$ has large λ-monodromy. $\square$

APPENDIX A. ARTIN-SCHREIER EQUATIONS

We establish a criterion for Artin-Schreier equations to be irreducible. We use this to calculate our Galois group.

Let $q = p^s$, and let $G \subset (\mathbb{F}_q, +)$ be a subgroup of order p^N. Define

$$f_G(X) = \prod_{\alpha \in G} (X - \alpha).$$

Since $f_G(X + \beta) = f_G(X)$ for $\beta \in H$, $f_G(X) = \sum_{j=0} N a_j X^{p^j}$. In particular, f_G is additive.

Lemma A.1. *Let K be any field containing an $\mathbb{F}_q$, and let $A \in K$. Then*

$$F(X) = X^q - X - A$$

is reducible if and only if $A = f_G(a)$ for some $a \in K$ and some nontrivial subgroup $G \subseteq \mathbb{F}_q^+$.

Proof. We write the polynomial as

$$F(X) = X^q - X - A = f_{\mathbb{F}_q}(X) - A.$$

Assume $F = \prod f_i$ is a non-trivial factorization of F into irreducible monic factors. Let y_1 denote a root of f_1. The roots of F are $y_1 + \beta, \beta \in \mathbb{F}_q$. Thus once we adjoin y_1 to K we can split all the f_i and thus the splitting fields of the f_i are all the same. There is a subgroup H of $\mathbb{F}_q$ so that $f_1(X) = \prod_{\alpha \in H}(X - (y_1 + \alpha))$. Since $f_H(X) - f_H(y_1)$ vanishes on the set $y_1 + H$ and has leading coefficient 1, we have $f_1(X) = f_H(X) - f_H(y_1)$. For $\beta \in \mathbb{F}_q$, let $[\beta] = \beta + H$ denote the corresponding equivalence class in $\mathbb{F}_q/H$. Define

$$f_{[\beta]}(X) = f_\beta(X) = f_1(X - \beta).$$

This is independent of the choice of β in $[\beta]$. Since the set of roots of f_β is $y_1 + \beta + H$, we have $F = \prod_{[\beta] \in \mathbb{F}_q/H} f_{[\beta]}$. The constant term of $f_\beta = f_1(X - \beta) = f_H(X - \beta) - f_H(y_1)$ is

$$f_H(-\beta) - f_H(y_1) = -(f_H(\beta) + f_H(y_1)).$$

Thus

$$A = \prod_{[\beta] \in \mathbb{F}_q/H} (-(f_H(\beta) + f_H(y_1))).$$

The map f_H is additive, maps $\mathbb{F}_q$ to itself, and has kernel exactly H. Let L denote the image of $\mathbb{F}_q$ under f_H. Then we can write

$$A = f_L(-f_H(y_1)).$$

Since $f_H(y_1)$ is the constant term in f_1, it is in K. □

Lemma A.2. *Let $F(X) = X^{p^n} + c_{n-1}X^{p^{n-1}} + \cdots + c_0 X$ be an additive polynomial with coefficients in a finite field. Let k be any field containing the coefficients of F, and let $K = k((z_1, \cdots z_e))((t))$. Suppose*

$$A = z_1^M(d_{-N}t^{-N} + d_{-N+1}t^{-N+1} + \cdots) + B \in K$$

where $M, N \in \mathbb{N}$, $d_i \in k$, $d_{-N} \neq 0$, and $B \in k((t))$. Then $F(X) = A + B$ has no solution in K.

Proof. Possibly after enlarging k, we may and do assume that $d_{-N} = 1$. We define an endomorphism π of K considered as k-vector space. Let $j \in \mathbb{Z}$, and $\alpha \in \mathbb{Z}^e$. Let

$$\pi_j(z^\alpha) = \begin{cases} z^\alpha & \text{if } \alpha = (i, 0 \cdots 0) \text{and } \frac{i}{j} = \frac{M}{-N} \\ 0 & \text{otherwise} \end{cases},$$

Now extend π_j to Laurent series by setting $\pi_j(\sum_\alpha a_\alpha z^\alpha) = \sum_\alpha a_\alpha \pi_j(z^\alpha)$. For $x = \sum_j A_j t^j$ $A_j \in k((z_1, \cdots, z_e))$, define $\pi(x) = \sum \pi_j(A_j) t^j$.

Then π is an idempotent operator; $\pi \circ \pi = \pi$. Moreover, π commutes with the p^{th}-power map; for $\alpha \in K$, $\pi(\alpha^p) = (\pi(\alpha))^p$. In particular, π commutes with F as k-linear endomorphisms of K.

This implies that if $F(X) = A + B$ has a solution in $x \in K$, then $\pi(x)$ is a solution to $F(X) = \pi(A + B) = \pi(A)$. Indeed, if $F(x) = A + B$, then $F(\pi(x)) = \pi(F(x)) = \pi(A + B)$. Since $\pi(A + B) = \pi(A) = z^M t^{-N}$, we are reduced to showing that $F(X) = z^M t^{-N}$ has no solution of the form $\pi(x)$ for $x \in K$.

Write $e = \gcd(M, N)$, $m = M/e$, $n = N/e$, and $w = z_1^m t^{-n}$. The k-linear space $\pi(K)$ can be identified with $k((w^{-1}))$. We are reduced to showing that

$$F(X) = w^e$$

has no solution in $k((w^{-1}))$. Since F maps each of $k[w]$ and $k[\![w^{-1}]\!]$ to itself, we are further reduced to showing that $F(X) = w^e$ has no solution in $k[w]$. But $F(x)$ never produces a monomial for any $x \in k[w]$ of positive degree. Thus the equation $F(x) = w^e$ has no solution. □

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